VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

SYNNO'S SOL[ns].

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Victorian Certificate of Education
2023

STUDENT NUMBER | | | | | | | | | | Letter

MATHEMATICAL METHODS
Written examination 1

Wednesday 1 November 2023

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

Note: The exams are Scanned for Marking Use a dark pen or 2B pencil if you must use pencil.

Note: It is risky to use erasal pen. If expose to heat the text may discepear

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 14 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

DO NOT WRITE IN THIS AREA

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

a. Let $y = \dfrac{x^2 - x}{e^x}$.

$u = x^2 - x \quad \frac{du}{dx} = 2x - 1$ Quotient Rule.

$v = e^x \quad \frac{dv}{dx} = e^x$

Find and simplify $\dfrac{dy}{dx}$. 2 marks

$$\frac{e^x \times (2x-1) - (x^2 - x)e^x}{(e^x)^2} = \frac{\cancel{e^x}\left[(2x-1) - (x^2 - x)\right]}{\cancel{e^x}\, e^x}$$

$$= \frac{-x^2 + 3x - 1}{e^x} = \frac{-(x^2 - 3x + 1)}{e^x}$$

b. Let $f(x) = \sin(x)e^{2x}$.

$u = \sin(x) \quad \frac{du}{dx} = \cos(x)$ Product Rule.

Find $f'\left(\dfrac{\pi}{4}\right)$. 2 marks

$v = e^{2x} \quad \frac{dv}{dx} = 2e^{2x}$

$$f'(x) = \sin(x) \times 2e^{2x} + \cos(x)e^{2x}$$

$$f'\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) \times 2e^{2\times\frac{\pi}{4}} + \cos\left(\frac{\pi}{4}\right)e^{2\times\frac{\pi}{4}}$$

$$= \cancel{2} \times \frac{\sqrt{2}}{\cancel{2}} \times e^{\frac{\pi}{2}} + \frac{\sqrt{2}}{2}e^{\frac{\pi}{2}}.$$

$$= \sqrt{2}\,e^{\frac{\pi}{2}} + \frac{\sqrt{2}}{2}e^{\frac{\pi}{2}}.$$

$$= \frac{3\sqrt{2}}{2}e^{\frac{\pi}{2}}$$

Question 2 (3 marks)

Solve $e^{2x} - 12 = 4e^x$ for $x \in R$.

$e^{2x} - 4e^{x} - 12 = 0$

Let $A = e^{x}$

$A^2 - 4A - 12 = 0$

$(A-6)(A+2) = 0$

$A - 6 = 0$ or $A + 2 = 0$

$A = 6 \quad A = -2$

$e^{x} = 6 \qquad e^{x} = -2$

$\log_e(e^{x}) = \log_e(6) \qquad$ No Sol^n

$x = \log_e(6)$

$= \log_e(3 \times 2)$

$= \log_e(3) + \log_e(2)$.

Question 3 (4 marks)

Ensure you label what has been asked for.

a. Sketch the graph of $f(x) = 2 - \dfrac{3}{x-1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates. 3 marks

$x - 1 \neq 0$

$x \neq 1$

$\Rightarrow$ Asymptote $x = 1$.

y-int $x = 0$.

$f(0) = 2 - \frac{3}{0-1}$

$= 2 + 3$

$= 5$

x-int $y = 0$

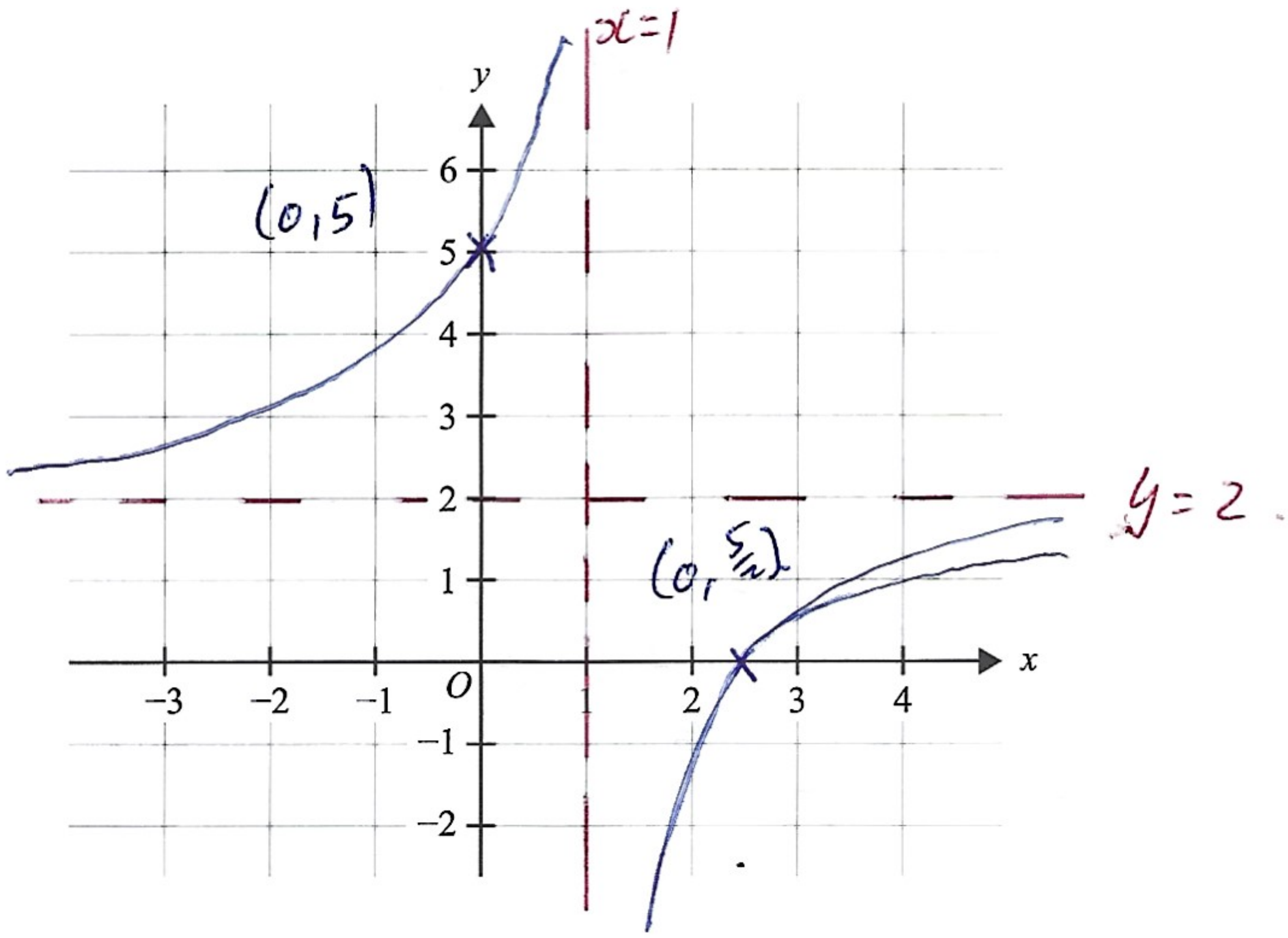
$0 = 2 - \frac{3}{x-1}$

$\frac{3}{x-1} = 2$

$3 = 2x - 2$

$5 = 2x$

$x = \frac{5}{2}$.



b. Find the values of x for which $f(x) \leq 1$. 1 mark

$f(x) = 1$

$1 = 2 - \frac{3}{x-1}$

$-1 = -\frac{3}{x-1}$

$1 = \frac{3}{x-1}$

$x - 1 = 3$

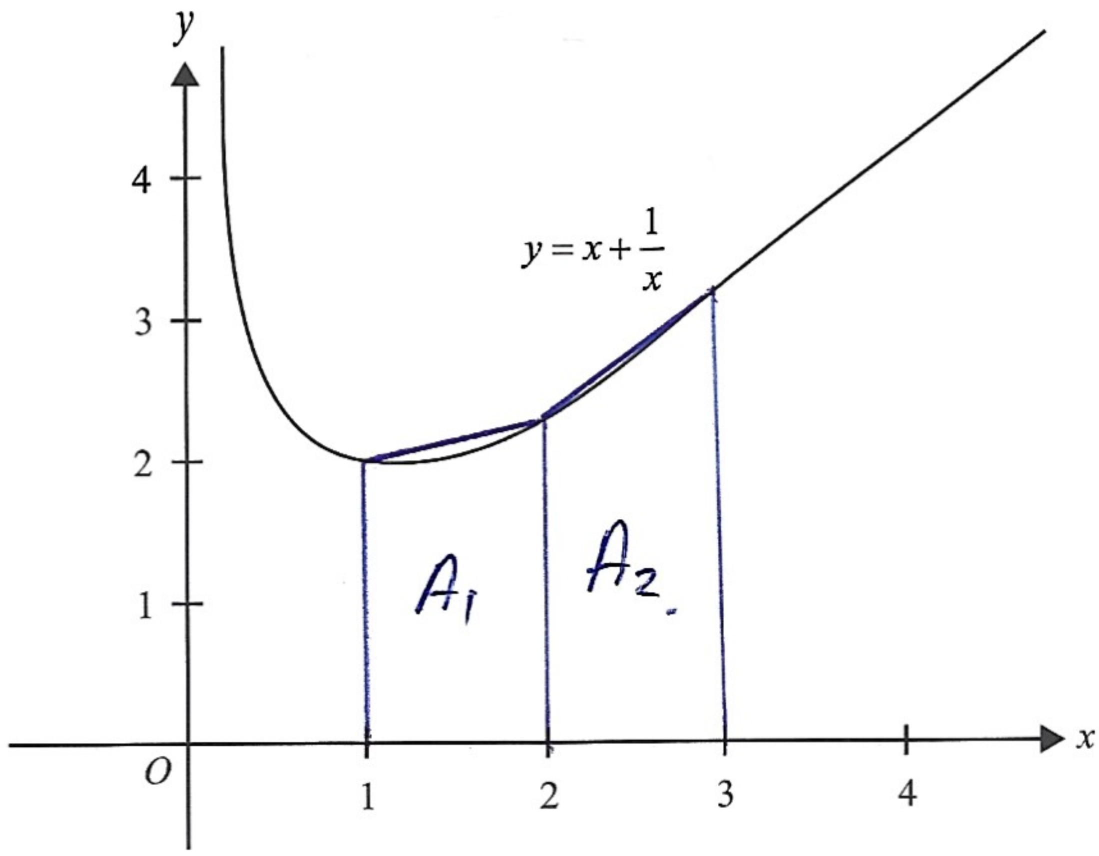
$x = 4$

$(1, 4]$

From Graph.

Question 4 (2 marks)

The graph of $y = x + \frac{1}{x}$ is shown over part of its domain.



Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

When $x = 1 \quad y = 1 + \frac{1}{1} = 2.$

$x = 2 \quad y = 2 + \frac{1}{2} = \frac{5}{2}.$

$x = 3 \quad y = 3 + \frac{1}{3} = \frac{10}{3}$

$A_1 = \frac{1}{2}\left(2 + \frac{5}{2}\right) \times 1 = \frac{1}{2}\left(\frac{9}{2}\right) \times 1 = \frac{9}{4}$

$A_2 = \frac{1}{2}\left(\frac{5}{2} + \frac{10}{3}\right) \times 1 = \frac{1}{2}\left(\frac{15}{6} + \frac{20}{6}\right) \times 1 = \frac{1}{2}\left(\frac{35}{6}\right) = \frac{35}{12}.$

Area $= A_1 + A_2$

Area $= \frac{9}{4} + \frac{35}{12}$

$= \frac{27}{12} + \frac{35}{12}$

$= \frac{62}{12}$

$= \frac{31}{6}.$

Note: Formula on Formula sheet for Area of Trapezium. Must use that as the question asks you to use Trapezium

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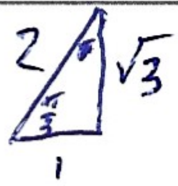
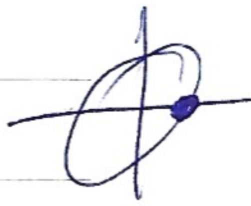
Question 5 (4 marks)

a. Evaluate $\int_0^{\frac{\pi}{3}} \sin(x)dx$. 1 mark

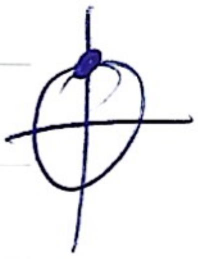


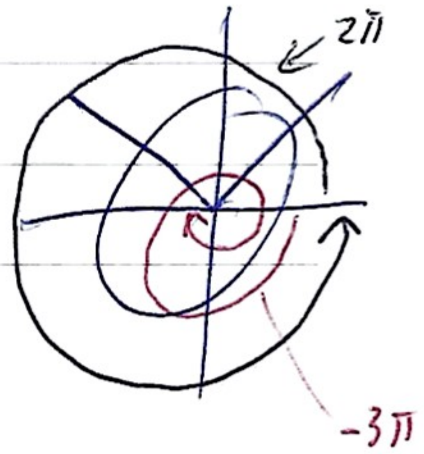
$$= \left[-\cos(x)\right]_0^{\pi/3}$$
$$= -\cos\left(\tfrac{\pi}{3}\right) - {}^{-}\cos(0)$$
$$= -\tfrac{1}{2} + 1$$
$$= \tfrac{1}{2}.$$



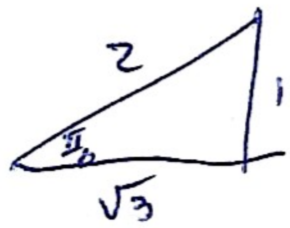
b. Hence, or otherwise, find all values of k such that $\int_0^{\frac{\pi}{3}} \sin(x)dx = \int_k^{\frac{\pi}{2}} \cos(x)dx$, where $-3\pi < k < 2\pi$. 3 marks

$$\tfrac{1}{2} = \int_k^{\frac{\pi}{2}} \cos(x)\,dx$$
$$\tfrac{1}{2} = \left[\sin(x)\right]_k^{\frac{\pi}{2}}$$
$$\tfrac{1}{2} = \sin\left(\tfrac{\pi}{2}\right) - \sin(k)$$
$$\tfrac{1}{2} = 1 - \sin(k).$$
$$-\tfrac{1}{2} = -\sin(k)$$
$$\sin(k) = \tfrac{1}{2}.$$





$$k = -2\pi + \tfrac{\pi}{6},\ -\pi - \tfrac{\pi}{6},\ \tfrac{\pi}{6},\ \pi - \tfrac{\pi}{6}$$
$$= -\tfrac{11\pi}{6},\ -\tfrac{7\pi}{6},\ \tfrac{\pi}{6},\ \tfrac{5\pi}{6}.$$



$\sin(k)$ +ve $\Rightarrow$ 1st & 2nd Quad.

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $\Rightarrow$ Reference Angle $= \frac{\pi}{6}$.

Question 6 (4 marks)

$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

Let $\hat{P}$ be the random variable that represents the sample proportion of households in a given suburb that have solar panels installed.

From a sample of randomly selected households in a given suburb, an approximate 95% confidence interval for the proportion p of households having solar panels installed was determined to be (0.04, 0.16).

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

$\hat{p} = \frac{0.04+0.16}{2} = \frac{0.2}{6} = 0.1$

Use $z = 2$ to approximate the 95% confidence interval.

b. Find the size of the sample from which this 95% confidence interval was obtained. 2 marks

$0.04 = \hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$0.04 = 0.1 - 2 \times \sqrt{\frac{0.1\times(1-0.1)}{n}}$

$-0.06 = -2 \times \sqrt{\frac{0.1\times 0.9}{n}}$

$0.03 = \sqrt{\frac{0.09}{n}}$

$0.0009 = \frac{0.09}{n}$

$n = \frac{0.09}{0.0009}$

$= 100$

c. A larger sample of households is selected, with a sample size four times the original sample. The sample proportion of households having solar panels installed is found to be the same.

By what factor will the increased sample size affect the width of the confidence interval? 1 mark

$n \rightarrow 4n$

$z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \rightarrow 2\sqrt{\frac{0.09}{4n}} \rightarrow \frac{2}{2}\sqrt{\frac{0.09}{n}}$

$\rightarrow \sqrt{\frac{0.09}{n}}$

$\therefore$ Width of confidence interval now half of what it was.

Note: There are other ways of looking at it.
In essence sd is $\sqrt{\frac{1}{4}} = \frac{1}{2}$ what it was before.
$\Rightarrow$ width now $\frac{1}{2}$ of before.

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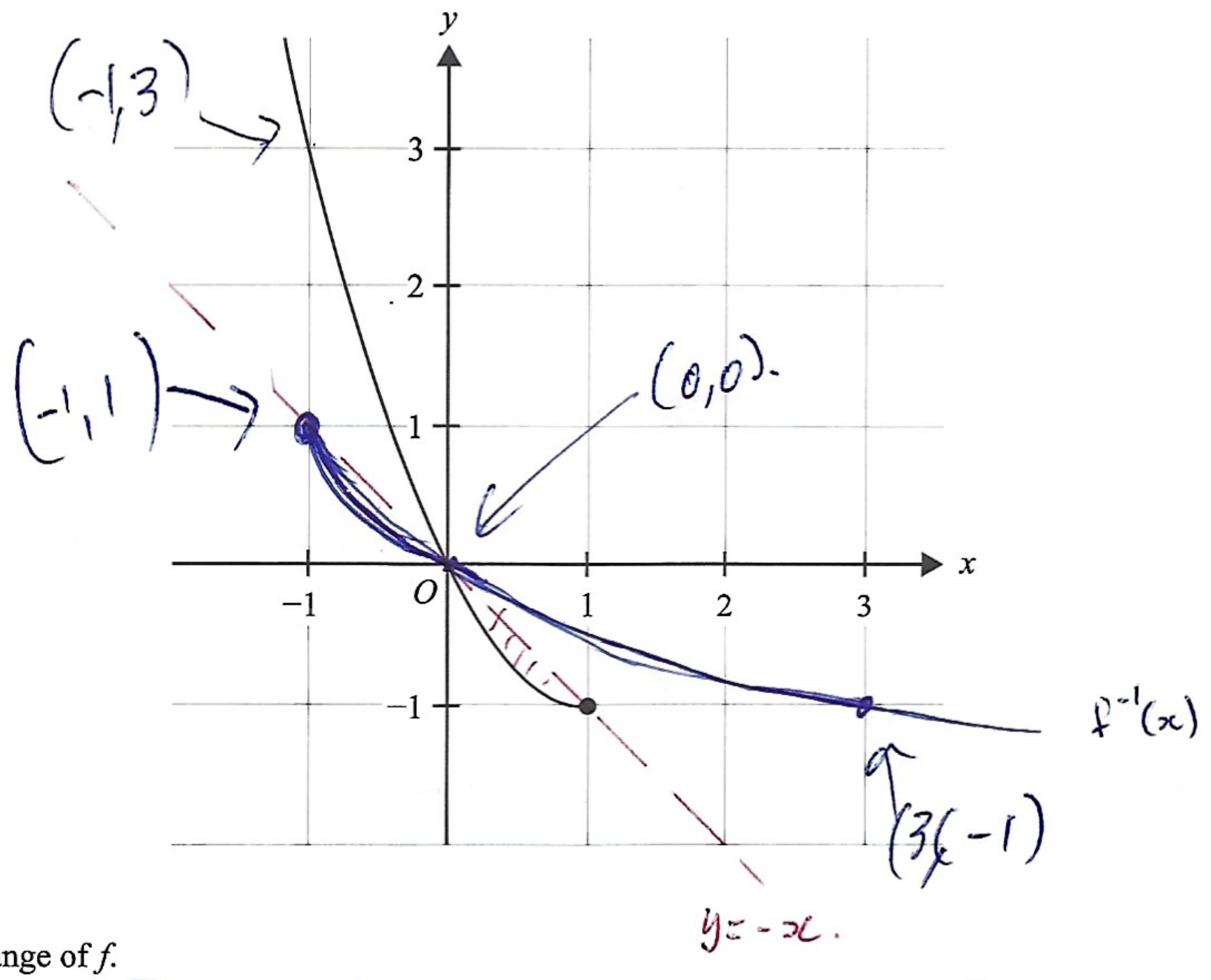
Question 7 (7 marks) Make Sure you label what they want.

Consider $f: (-\infty, 1] \rightarrow R, f(x) = x^2 - 2x$. Part of the graph of $y = f(x)$ is shown below.



a. State the range of f. 1 mark

$[-1, \infty)$

Note: $[\infty, -1]$ is incorrect as it does not follow the accepted convention.

b. Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes above. Label any endpoints and axial intercepts with their coordinates. 2 marks

c. Determine the equation and the domain for the inverse function f^{-1}. 2 marks

$x = (y^2 - 2y + 1) - 1$

$x = (y-1)^2 - 1$

$x + 1 = (y-1)^2$

$\pm\sqrt{x+1} = y - 1$

we want $-\sqrt{x+1} = y - 1$

$y = 1 - \sqrt{x+1}$

Note: the use of y is for our convienience

$f^{-1}(x) = 1 - \sqrt{x+1}$

domain $[-1, \infty)$.

Muse give final answer as $f^{-1}(x)$

Question 7 – continued

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$. 2 marks

$$\text{Area} = \left| 2 \times \left[\int_0^1 (x^2 - 2x)\,dx - \int_0^1 (-x)\,dx \right] \right|$$

$$= \left| 2 \times \left[\left[\frac{x^3}{3} - x^2 \right]_0^1 - \left[-\frac{x^2}{2} \right]_0^1 \right] \right|$$

$$= \left| 2 \times \left[\left[\left(\frac{1}{3} - 1\right) - (0 - 0) \right] - \left[-\frac{1}{2} - 0 \right] \right] \right|$$

$$= \left| 2 \times \left[\left[-\frac{2}{3} \right] + \frac{1}{2} \right] \right|$$

$$= \left| 2 \times \left[-\frac{1}{6} \right] \right|$$

$$-\frac{4}{6} + \frac{3}{6}$$
$$-\frac{1}{6}.$$

$$= \left| -\frac{1}{3} \right|$$

$$= \frac{1}{3}$$

Note: There are other ways of setting out / looking at part d.

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Question 8 (6 marks)

Suppose that the queuing time, T (in minutes), at a customer service desk has a probability density function given by

Area = 1

$$f(t) = \begin{cases} kt(16 - t^2) & 0 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some $k \in R$.

a. Show that $k = \dfrac{1}{64}$. 1 mark

$\int_0^4 kt(16-t^2)\,dt = 1$

$k\int_0^4 (16t - t^3)\,dt = 1$

$k\left[8t^2 - \frac{t^4}{4}\right]_0^4 = 1$

$k\left[\left(8\times4^2 - \frac{4^4}{4}\right) - (0-0)\right] = 1$

$k\left[(8\times16 - 4^3)\right] = 1$

$k[128 - 64] = 1$ $\quad k\times 64 = 1$ $\quad k = \frac{1}{64}$

b. Find E(T). 2 marks

$E(T) = \int_0^4 t\times kt(16-t^2)\,dt$

$= \frac{1}{64}\int_0^4 (16t^2 - t^4)\,dt$

$= \frac{1}{64}\left[\frac{16t^3}{3} - \frac{t^5}{5}\right]_0^4$

$= \frac{1}{64}\left[\left(\frac{16\times4^3}{3} - \frac{4^5}{5}\right) - \left(\frac{0}{3} - \frac{0}{5}\right)\right]$

$= \frac{1}{64}\left[\frac{1024}{3} - \frac{1024}{5}\right]$

$= \frac{1}{64}\left[\frac{5\times1024 - 3\times1024}{15}\right]$

$= \frac{1}{\cancel{64}_{1}}\left[\frac{2\times\cancel{1024}^{16}}{15}\right]$

$= \frac{32}{15}$

Note: This was messy. See next page for arithmetic. Possibly leaving in terms of 4^3, 4^5 etc would have been better

Question 8 – continued

c. What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute? 3 marks

$$Pr(T>2 \mid T>1) = \frac{Pr(T>2)}{Pr(>1)}$$

$$= \frac{\frac{1}{64}\int_2^4 (16t - t^3)\,dt}{\frac{1}{64}\int_1^4 (16t - t^3)\,dt}$$

$$= \frac{\left[8t^2 - \frac{t^4}{4}\right]_2^4}{\left[8t^2 - \frac{t^4}{4}\right]_1^4}$$

$$= \frac{\left(8\times 4^2 - \frac{4^4}{4}\right) - \left(8\times 2^2 - \frac{2^4}{4}\right)}{\left(8\times 4^2 - \frac{4^4}{4}\right) - \left(8\times 1^2 - \frac{1^4}{4}\right)}$$

$$= \frac{(128 - 64) - (32 - 4)}{(128 - 64) - \left(8 - \frac{1}{4}\right)}$$

$$= \frac{64 - 28}{64 - \frac{31}{4}}$$

$$= \frac{36}{\frac{256 - 31}{4}}$$

$$= \frac{36}{\frac{225}{4}}$$

$$= 36 \times \frac{4}{225} = \frac{144}{225} = \frac{16}{25}$$

Arithmetic for Part b.

$$16 \times 8 = 128 \qquad 16 \times 4 = 64$$

$$128 - 64 = 64 \qquad 64 \times 4 = 256 \qquad 256 \times 4 = 1024$$

$4^3 = 64$

$4^4 = 256.$

$4^5 = 1024$

$$64 \times 16: \quad 384 + 640 = 1024$$

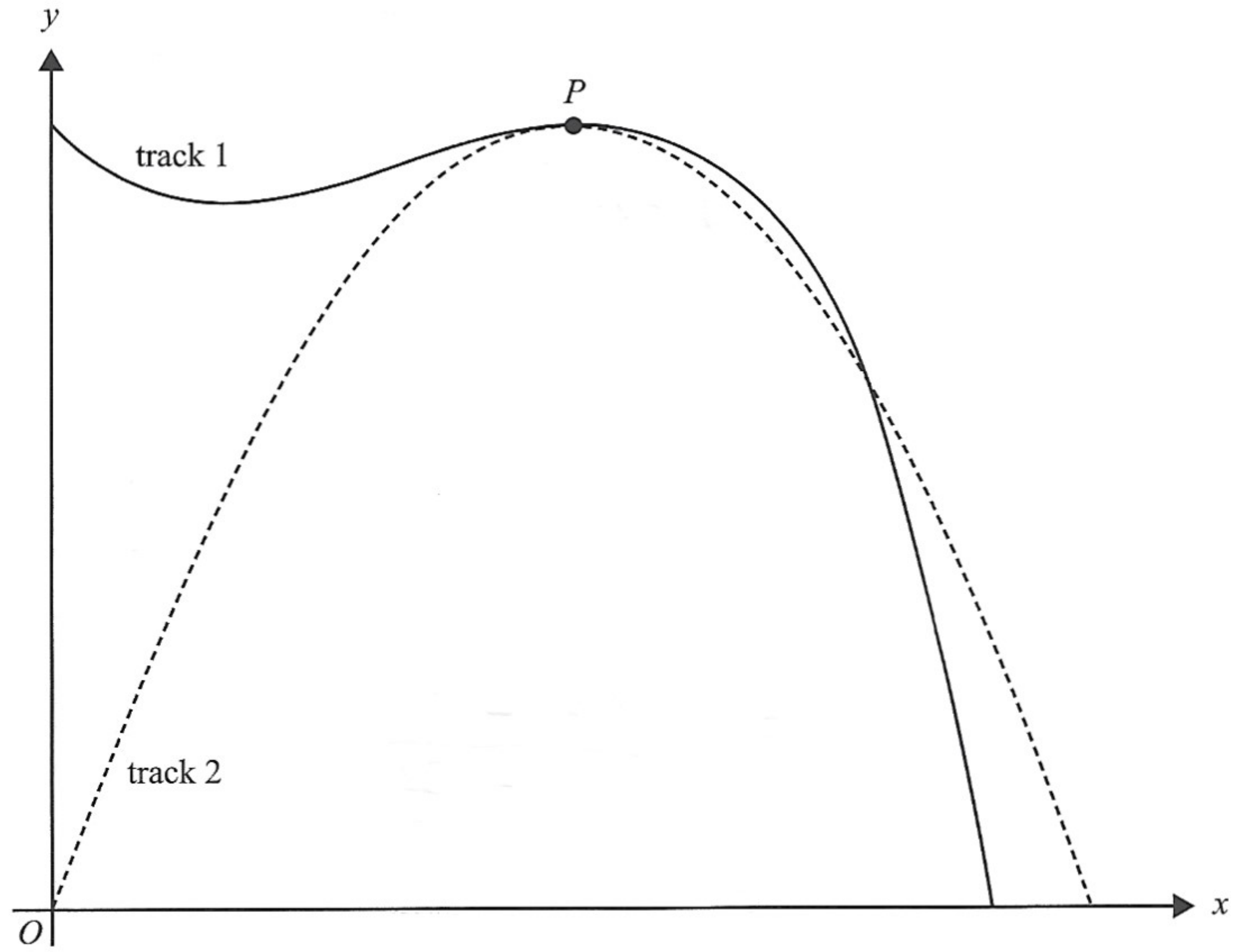
$$84 - 28 = 36$$

$$256 - 31 = 225$$

$$36 \times 4 = 144$$

Question 9 (6 marks)

The shapes of two walking tracks are shown below.



Track 1 is described by the function $f(x) = a - x(x-2)^2$.

Track 2 is defined by the function $g(x) = 12x + bx^2$.

The unit of length is kilometres.

a. Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$. 1 mark

$f(0)$ $12 = a - 0(0-2)^2$

$12 = a - 0$

$a = 12$.

$g(1)$ $9 = 12\times 1 + b \times 1^2$

$9 = 12 + b$

$b = -3$.

Note: Verify $\Rightarrow$ Show calculation.

b. Verify that $f(x)$ and $g(x)$ both have a turning point at P.
Give the co-ordinates of P.

2 marks

$f(x) = 12 - x(x-2)^2$

$f(x) = 12 - x(x^2 - 4x + 4)$

$f(x) = 12 - x^3 + 4x^2 - 4x$

$f'(x) = -3x^2 + 8x - 4$

$f'(x) = -3\left[\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + \frac{4}{3} - \frac{16}{9}\right]$

$f'(x) = -3\left[\left(x - \frac{4}{3}\right)^2 - \frac{4}{9}\right]$

$0 = -3\left[\left(x - \frac{4}{3}\right)^2 - \frac{4}{9}\right]$

$0 = \left(x - \frac{4}{3}\right)^2 - \frac{4}{9}$

$\frac{4}{9} = \left(x - \frac{4}{3}\right)^2$

$\frac{2}{3} = x - \frac{4}{3}$

$x = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2.$

$f(2) = 12 - 2(2-2)^2$

$= 12 - 2 \times (0)^2$

$= 12.$

T.P. at $(2, 12)$

$g(x) = 12x - 3x^2$

$g'(x) = 12 - 6x$

$0 = 12 - 6x$

$6x = 12$

$x = 2.$

$g(2) = 12 \times 2 - 3 \times 2^2$

$= 24 - 12$

$= 12.$

T.P. at $(2, 12)$.

Co-ordinates of P $(2, 12)$

$\frac{4}{3} - \frac{16}{9}$

$\frac{12}{9} - \frac{16}{9}$

$-\frac{4}{9}$

Note: Messy way of doing $f'(x)$
Could do

$0 = -3x^2 + 8x - 4$

$0 = 3x^2 - 8x + 4$

$0 = (3x-2)(x-2)$

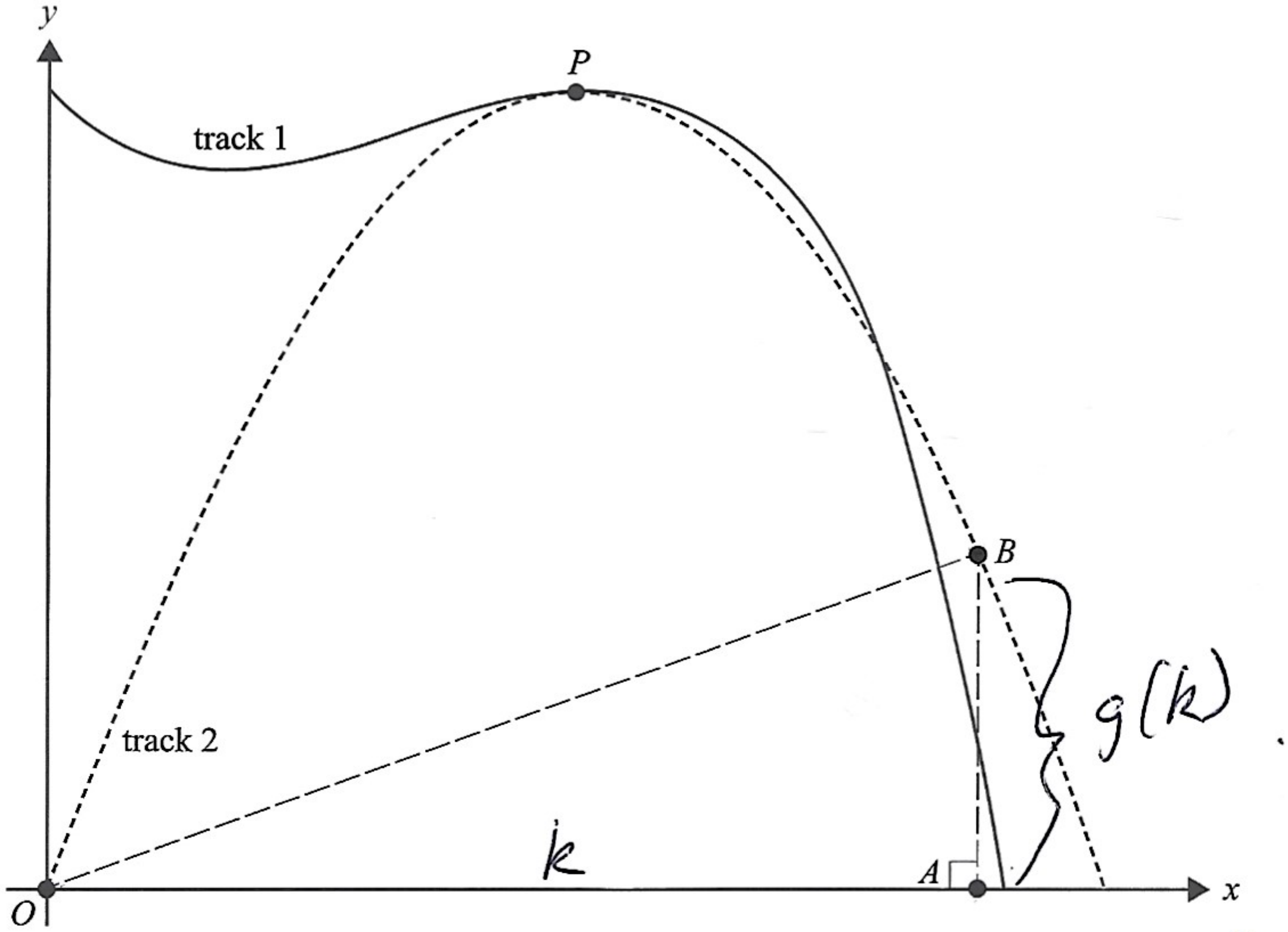
$\vdots$

w

which is less fiddly.

Question 9 – continued
TURN OVER

c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $(k, g(k))$, as shown below, where $k \in (0, 4)$.
Find the maximum possible area of the theme park, in km^2. 3 marks



$$\text{Area} = \frac{1}{2} \times \overline{OA} \times \overline{AB}$$
$$= \frac{1}{2} \times k \times g(k)$$
$$= \frac{1}{2} \times k \times (12 \times k - 3k^2)$$
$$= \frac{1}{2} \times k \times (12k - 3k^2)$$
$$\text{Area} = 6k^2 - \frac{3}{2}k^3$$
Max Area when $\frac{d\,\text{Area}}{dk} = 0$
$$12k - \frac{9}{2}k^2 = 0$$
$$k\left(12 - \frac{9}{2}k\right) = 0$$
$$k = 0 \qquad 12 - \frac{9}{2}k = 0$$
$$\frac{9}{2}k = 12$$
$$k = \frac{24}{9}$$
$$= \frac{8}{3}$$

$$\text{Area} = 6\left(\frac{8}{3}\right)^2 - \frac{3}{2}\left(\frac{8}{3}\right)^3$$
$$= \frac{\cancel{6}^{2} \times 64}{\cancel{9}_{3}} - \frac{\cancel{3}^{1}}{\cancel{2}_{1}} \times \frac{\cancel{512}^{256}}{\cancel{27}_{9}}$$
$$= \frac{128}{3} - \frac{256}{9}$$
$$= \frac{384}{9} - \frac{256}{9}$$
$$= \frac{128}{9}$$

END OF QUESTION AND ANSWER BOOK

$$\begin{array}{r} 64 \\ 8 \\ \hline 512 \end{array}$$

$$2\overline{)512}\;\; 256$$

~~9) 256~~

~~3) 256~~

$$\begin{array}{r} 128 \\ \times\quad 3 \\ \hline 384 \end{array}$$

$$\begin{array}{r} 384 \\ -256 \\ \hline 128 \end{array}$$